investigations will be necessary to resolve the questions involved.

Turning to the local heat flux, Figs. 11–13 are summaries of the distributions over the spherical portion of the model. The glass model data and the nickel model data agree well up to $\theta \approx 80°$. The data scatter becomes progressively worse as θ approaches 90°, since the heat flux becomes quite small and experimental uncertainties become progressively more important. The local heat flux agrees well with Lees' theory for θ less than 45° and does not show any substantial Reynolds number dependence. The Mach 2 data agree with Lees' theory over the entire hemisphere; however, the Mach 4 and 6 data are somewhat higher than Lees' predictions for θ greater than 45°. This deviation corresponds to the pressure coefficient's deviation from the Newtonian distribution in these areas. It is surprising that the Mach 2 heat transfer data and pressure distribution data support a Newtonian distribution better than the Mach 4 and 6 data.

References

[1] Probstein, R. F., "Shock wave and flow field development in hypersonic re-entry," ARS J. **31**, 185–194 (1961).

[2] Hayes, W. D. and Probstein, R. F., *Hypersonic Flow Theory* (Academic Press, New York, 1959), p. 372.

[3] Hoshizaki, H., "Shock generated vorticity at low Reynolds numbers," Lockheed Missiles & Space Div. Rept. 48381, Vol. 1, pp. 9–43 (1959).

[4] Ho, H. T. and Probstein, R. F., *Rarefied Gas Dynamics* (Academic Press, New York, 1961), pp. 525–552.

[5] Cheng, H. K., "Hypersonic shock-layer theory of the stagnation region at low Reynolds number," *Proceedings of the 1961 Heat Transfer and Fluid Mechanics Institute* (Stanford University Press, Stanford, Calif., 1962), pp. 161–175.

[6] Ferri, A., Zakkay, V., and Ting, L., "Blunt body heat transfer at hypersonic speed and low Reynolds numbers," Polytech. Inst. Brooklyn, PIBAL Rept. 611 (1960); also J. Aerospace Sci. **28**, 962–971 (1961).

[7] Van Dyke, M., "Second-order compressible boundary-layer theory with application to blunt bodies in hypersonic flow," Stanford Univ., Aeronaut. Eng. Div. Rept. AFOSR-TN-61-1270 (1961).

[8] Ferri, A. and Zakkay, V., "Measurements of stagnation point heat transfer at low Reynolds numbers," Polytech. Inst. Brooklyn, ARL TR 38 (1961).

[9] Frössling, N., "Mass transfer, heat transfer and skin friction in two-dimensional and rotationally symmetrical laminar boundary layers," Natl. Advisory Committee for Aeronaut. Rept. TN-1462 (1958).

[10] Sibulkin, M. J., "Heat transfer near the forward stagnation point of a body of revolution," J. Aeronaut. Sci. **19**, 570–571 (1952).

[11] Fay, J. A. and Riddell, F. A., "Theory of stagnation point heat transfer in dissociated air," J. Aeronaut. Sci. **25**, 73–85 (1958).

[12] Lees, L., "Laminar heat transfer over blunt-nosed bodies at hypersonic flight speeds," Jet Propulsion **26**, 259–269, 274 (1956).

[13] Neice, S. E., Rutkowski, R. W., and Chan, K. K., "Stagnation-point heat-transfer measurements in hypersonic, low-density flow," J. Aerospace Sci. **27**, 387–388 (1960).

[14] Wilson, M. R. and Wittliff, C. E., "Low density stagnation point heat transfer measurements in the hypersonic shock tunnel," ARS J. **32**, 275–276 (1962).

[15] Maslach, G. J. and Sherman, F. S., "Design and testing of an axisymmetric hypersonic nozzle for a low density wind tunnel," Univ. Calif., Wright Air Dev. Div. TR 341 (1956).

[16] Owen, J. M. and Sherman, F. S., "Design, fabrication, and evaluation of a Mach 4 axially symmetric nozzle for rarefied gas flows," Univ. Calif., Eng. Project Rept. HE-150-104 (1952).

[17] Tewfik, O. K. and Giedt, W. H., "Heat transfer, recovery factor, and pressure distributions around a circular cylinder normal to a supersonic rarefied air stream," J. Aeronaut. Sci. **27**, 721–729 (1960).

[18] Hickman, R. S., "The influence of shock wave-boundary layer interaction on heat transfer to an axisymmetric body," Univ. Calif., Eng. Project Rept. HE-150-191 (1960).

[19] Hilsenrath, J., "Tables of thermal properties of gases," Natl. Bur. Standards Circular 564 (1956).

MARCH 1963 AIAA JOURNAL VOL. 1, NO. 3

An Integral Method for Calculation of Supersonic Laminar Boundary Layer with Heat Transfer on Yawed Cone

SHEE-MANG YEN* AND N. A. THYSON†

University of Illinois, Urbana, Ill.

An integral method for calculating the three-dimensional boundary layer over the surface of a cone at angle of attack is investigated. The numerical procedure of integration for that method on the basis of a simplifying assumption concerning the boundary layer development along the cone generator is developed and illustrated by applying the method to find the solutions of integral equations for a specific example. The results obtained for the example for the range of circumferential angle of 40° investigated are summarized and given as heat transfer coefficients, coefficients of friction, and other friction parameters. The distribution of heat transfer coefficients checked with available experimental data fairly well.

Nomenclature

a = shape parameter in longitudinal velocity profile
b = shape parameter in cross flow velocity profile
c = shape parameter in enthalpy profile
C = parameter in Chapman-Rubesin temperature-viscosity relation
c_f = coefficient of friction
c_p = specific heat at constant pressure
f_w'' = shear parameter
h = heat transfer coefficient
H = total enthalpy
K = proportionality parameter in transformed boundary layer thickness
St = Stanton number
T = temperature
T_1 = dimensionless external temperature ($1/T_1 = u_e^2/2c_pT_e$)
u = velocity in x direction (along a generator)
v = velocity in y direction (perpendicular to surface of a cone)
w = velocity in circumferential direction
x = coordinate along generator

Received by IAS August 27, 1962; revision received January 7, 963.

* Associate Professor of Aeronautical and Astronautical Engineering.

† Formerly Graduate Student; now with Avco Corporation.

y = coordinate perpendicular to surface of cone
Y = transformed coordinate perpendicular to surface of cone
α = angle of attack
β = sine of cone semivertex angle
Δ = boundary layer thickness in transformed coordinates
η = dimensionless coordinates perpendicular to cone
θ = enthalpy difference ratio
ϕ = angular coordinate measured from stagnation line
ψ = angle between total coefficient of friction and that in x direction

Subscripts

e = outer edge of boundary layer
s = surface
t = total
0 = freestream stagnation
∞ = freestream
ϕ = at angle ϕ from the stagnation line

Superscript

* dimensionless quantity (with respect to values at the edge of the boundary layer)

Introduction

THE supersonic aerodynamics of pointed bodies is of interest in connection with the design of aircraft and missile fuselages. An important feature of the flow about such bodies is the behavior of the boundary layer and, in particular, the cross flow (flow in circumferential direction) that exists due to angle of attack. The characteristics of the three-dimensional boundary layer on the surface of a right circular cone at angle of attack to a supersonic stream are of interest. The flow around a cone at angle of attack can be divided into three types: 1) flow without separation when the angle of attack is less than the cone half-angle, 2) flow with separation when the angle of attack is greater than the cone half-angle but not great enough for the flow to approach that of a yawed cylinder, and 3) flow qualitatively the same as that about a yawed infinite cylinder when the angle of attack is very large. For large angles of attack, solutions are available only on the stagnation line.[1–4] Until now, no theoretical study has been made on the three-dimensional laminar boundary layer off the stagnation line associated with a cone at large angle of attack. There have been very few experimental investigations of the laminar boundary layer characteristics off the stagnation line, and only the circumferential distribution of heat transfer coefficients are available.[5–9] The present paper investigates an integral method to evaluate the boundary layer characteristics around a cone at angle of attack. The integral equations used are developed by using the method employed by Brunk,[2, 12] who investigated the boundary layer characteristics along the stagnation line. The velocity and enthalpy profiles in the boundary layer are approximated by three third-degree polynominals, each in terms of a single parameter. Blasius-type parabolic similarity is assumed to exist in the longitudinal direction.

Basic Equations

The following assumptions, as given in Ref. 2, are made in deriving the integral equations: 1) c_p = const, 2) $Pr = 1$, 3) the temperature-viscosity relation is that of Chapman-Rubesin, and 4) the surface temperature is constant.

Upon applying the boundary conditions, the velocity profiles and enthalpy profile that are approximated by three third-degree polynominals are

$$u^* = u/u_e = (3 - 2\eta)\eta^2 + a\eta(1 - 2\eta + \eta^2) \quad (1a)$$

$$w^* = w/u_e = (3 - 2\eta)\eta^2(w_e/u_e) + b\eta(1 - 2\eta + \eta^2) \quad (1b)$$

$$\theta^* = (H - H_s)/(H_0 - H_s) = (3 - 2\eta)\eta^2 + c\eta(1 - 2\eta + \eta^2) \quad (1c)$$

where

u = velocity in x direction (see Fig. 1)
v = velocity in y direction
w = velocity in circumferential direction
H = total enthalpy
η = Y/Δ
Y = $\int_0^y \rho^* \, dy$
Δ = $K(\mu_e/\rho_e u_e)^{1/2} x^{1/2}$

The expressions for the velocity and enthalpy profiles, Eqs. (1a–1c), and the expression for boundary layer thickness Δ then are substituted into the nondimensional incompressible form integral equations, and the final integral equations obtained are

$$\frac{27}{4} + \frac{9}{8}a - \frac{1}{2}a^2 + \frac{1}{\beta}\left[\frac{d}{d\phi}\ln K + \frac{d}{d\phi}\ln\rho_e u_e^2\right]\left[\frac{9}{2}w_e^* + \frac{11}{6}b - \frac{13}{12}aw_e - \frac{1}{3}ab\right] + 13w_e^{*2} + \frac{13}{6}bw_e^* + \frac{1}{3}b^2 + \frac{1}{\beta}\times$$
$$\left[\frac{9}{2}\frac{dw_e^*}{d\phi} + \frac{11}{6}\frac{db}{d\phi} - \frac{13}{12}a\frac{dw_e^*}{d\phi} - \frac{13}{12}w_e^*\frac{da}{d\phi} - \frac{1}{3}a\frac{db}{d\phi} - \frac{1}{3}b\frac{da}{d\phi}\right] - \frac{1}{\beta}\frac{d}{d\phi}\ln u_e\left[\frac{35}{2}w_e^* + \frac{35}{12}b\right] = 35\,a\frac{C}{K^2} \quad (2a)$$

$$-\frac{25}{4}w_e^* + \frac{5}{3}aw_e^* - \frac{65}{24}b - \frac{5}{6}ab + \frac{1}{\beta}\left[\frac{d}{d\phi}\ln K + \frac{d}{d\phi}\ln\rho_e u_e^2\right]\left[\frac{9}{2}w_e^{*2} + \frac{3}{4}bw_e^* - \frac{1}{3}b^2\right] + \frac{1}{\beta}\times$$
$$\left[9\,w_e^*\frac{dw_e^*}{d\phi} + \frac{3}{4}w_e^*\frac{db}{d\phi} + \frac{3}{4}b\frac{dw_e^*}{d\phi} - \frac{2}{3}b\frac{db}{d\phi}\right] - \frac{1}{\beta}\frac{dw_e^*}{d\phi}\left[\frac{35}{2}w_e^* + \frac{35}{12}b\right] + 35w_e^*\left[\frac{1}{\beta}\frac{dw_e^*}{d\phi} + 1\right]\times$$
$$\left\{1 + \frac{1}{T_1}\left(1 - \frac{13}{35} - \frac{13}{210}a - \frac{a^2}{105} - \frac{13}{35}w_e^{*2} - \frac{13}{210}bw_e^* - \frac{b^2}{105}\right) + \frac{1}{T_1}w_e^{*2} + \left[\frac{T_s}{T_0} - 1\right]\left[1 + \frac{1}{T_1}(1 + w_e^{*2})\right]\times\right.$$
$$\left.\left[\frac{1}{2} - \frac{c}{12}\right]\right\} = 35\,b\frac{C}{K^2} \quad (2b)$$

$$\frac{27}{4} + \frac{33}{12}a - \frac{39}{24}c - \frac{1}{2}ac + \frac{1}{\beta}\left[\frac{d}{d\phi}\ln K + \frac{d}{d\phi}\ln\rho_e u_e\right]\left[\frac{9}{2}w_e^* + \frac{11}{6}b - \frac{13}{12}cw_e^* - \frac{1}{3}bc\right] +$$
$$\frac{1}{\beta}\left[\frac{9}{2}\frac{dw_e^*}{d\phi} + \frac{11}{6}\frac{db}{d\phi} - \frac{13}{12}c\frac{dw_e^*}{d\phi} - \frac{13}{12}w_e^*\frac{dc}{d\phi} - \frac{1}{3}b\frac{dc}{d\phi} - \frac{1}{3}c\frac{db}{d\phi}\right] = 35\,c\frac{C}{K^2} \quad (2c)$$

Equations (2a) and (2b) are momentum equations in the x and ϕ directions, respectively, and Eq. (2c) is the energy equation. These three simultaneous equations are first-order, ordinary, nonlinear differential equations in four unknowns [$a = (du^*/d\eta)_0$, $b = (dw^*/d\eta)_0$, $c = (d\theta^*/d\eta)_0$, and K].

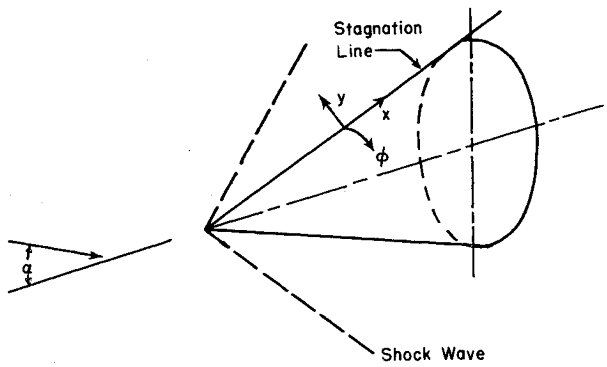

Fig. 1 Coordinate system for circular cone at angle of attack

Numerical Method of Solving the Integral Equations

If the three integral equations are to be used to determine a, b, and c (i.e., the shape parameters to determine the velocity and temperature profiles), the variation of K in the circumferential direction has to be given. With the absence of such knowledge concerning K, it is decided to consider the case with K assumed constant. This assumption, implying that the development of boundary layer thickness in the longitudinal direction off the stagnation line is the same as that along the stagnation line, may not be justified for flow along cone generators at appreciable circumferential angle from the stagnation line. The numerical method is outlined as follows:

1) Select the following conditions: M_∞, T_s/T_0 (surface temperature), θ_c (cone angle), and α (angle of attack).

2) The starting values of a, b, and c and their derivatives with respect to ϕ (i.e., the values at the stagnation line) are determined from Eqs. (2a–2c) modified for application at stagnation line.

3) Determine the external flow parameters from the Massachusetts Institute of Technology cone tables.[10, 11]

4) Solve the integral equations for a, b, and c by using a stepwise procedure of numerical integration. It was observed that, besides the cross flow variable b, the changes in the term on the right-hand side of each of the three integral equations also are the controlling factors in changing the value at each step of approximation of the values of the derivatives.

Example

The suitability of using the numerical method described in the foregoing to solve the differential equations can be ascertained best by applying the method to a specific example. The calculations are made up to circumferential angle

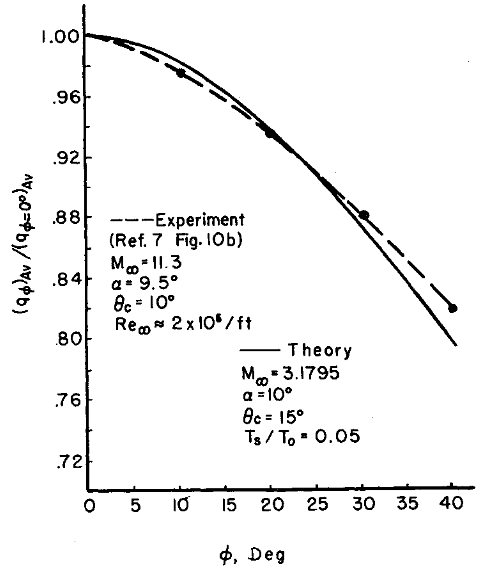

Fig. 2 Comparison of theoretical results and experimental data on heat transfer around a cone at angle of attack

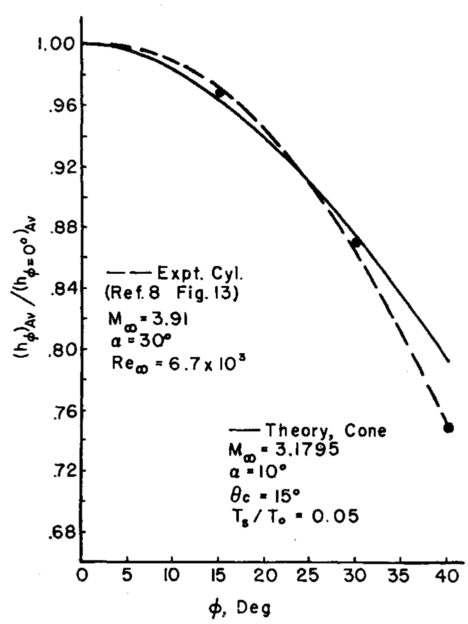

Fig. 3 Comparison of theoretical results on heat transfer coefficient around a cone at angle of attack and experimental data on heat transfer coefficient around a yawed cylinder

Table 1 Summary of results[a]

	a	b	c	$(c_{f,x})_\phi/(c_{f,x})_0$	$(c_{f,t})_\phi/(c_{f,t})_0$	f_w''	ψ	$(St)_\phi/(St)_0$	$(St)_\phi/(c_{f,x})_\phi$	$(St)_\phi/(c_{f,t})_\phi$
0°	1.5921	0	1.5921	1.0000	1.0000	0.5760	0°	1.0000	0.5000	0.5000
5°	1.5831	0.0747	1.5833	0.9948	0.9959	0.5730	2.70°	0.9951	0.4999	0.4996
10°	1.5784	0.1486	1.5781	0.9939	0.9982	0.5710	5.38°	0.9937	0.4999	0.4977
15°	1.5584	0.2306	1.5578	0.9802	0.9909	0.5630	8.24°	0.9799	0.4998	0.4944
20°	1.5001	0.2929	1.4999	0.9427	0.9605	0.5420	10.89°	0.9427	0.4997	0.4970
25°	1.4670	0.3513	1.4657	0.9223	0.9484	0.5290	13.47°	0.9216	0.4995	0.4859
30°	1.4441	0.4099	1.4418	0.9059	0.9417	0.5220	15.84°	0.9045	0.4992	0.4803
35°	1.3937	0.4618	1.3908	0.8733	0.9201	0.5010	18.33°	0.8716	0.4990	0.4737
40°	1.3344	0.5062	1.3307	0.8368	0.8950	0.4790	20.78°	0.8345	0.4986	0.4662

[a] Values of boundary layer parameters and variations of coefficients of friction and heat transfer coefficients around a cone for M_∞ = 3.1795, α = 10°, θ_c = 15°, T_s/T_0 = 0.05, and Pr = 1.0; $(c_{f,x})\phi$ = coefficient of friction in x direction at angle ϕ from the stagnation line; $(c_{f,t})\phi$ = total coefficient of friction at angle ϕ from the stagnation line.

of 40° for example of the following conditions: 1) $M_\infty = 3.1795$, 2) $T_s/T_0 = 0.05$, 3) $\theta_c = 15°$, and 4) $\alpha = 10°$. The external flow parameters used are based upon first-order theory. The results are shown in Table 1, which also includes the results of calculation on coefficient of friction, coefficient of friction in x direction, shear parameter f_w'', angle between total coefficient of friction and coefficient of friction in x direction, and Stanton number St. In the circumferential direction off the stagnation line (i.e., as ϕ increases) 1) the coefficient of friction in the x direction and f_w'' decrease, 2) the coefficient of friction in the x direction increases since it is directly proportional to b, 3) the total coefficient of friction decreases, and 4) the angle between the total coefficient of friction and that in the x direction increases. Comparisons are made of the distribution of heat transfer coefficient in the circumferential direction and that obtained by experiments. The present theoretical results are found to agree fairly well with the available experimental data for similar flow conditions. Two such comparisons are shown in Figs. 2 and 3. The experimental results in Fig. 2 are for much higher Mach number, and other flow parameters are slightly different.[7] The experimental results for Fig. 3 are for a yawed cylinder.

Discussion

The results on coefficients of friction and friction parameters are of special interest, since neither experimental data nor theoretical results on those are available. Any experimental data on coefficients of friction which may become available for a cone at angles of attack for comparison with the theoretical results obtained will be helpful in establishing the accuracy of the present method to investigate the complete boundary layer characteristics around a cone at angles of attack. The variation of K also should be considered in extending the calculation to the case of larger circumferential angles; however, it should be pointed out that the accuracy of the integral method suffers on the leeward side of the yawed cone which has an unfavorable pressure gradient. Furthermore, at large angles of attack, the boundary layer on the leeward side no longer can be considered as thin.

References

[1] Moore, F. K., "Laminar boundary-layer on a cone in supersonic flow at large angle of attack, "NACA TN 2844 (November 1952).

[2] Brunk, W. E., "Approximate method for calculation of laminar boundary-layer with heat-transfer on a cone at large angle of attack in supersonic flow," NACA TN 4380 (September 1958).

[3] Reshotko, E. and Beckwith, I. E., "Compressible laminar boundary-layer over a yawed infinite cylinder with heat-transfer and arbitrary Prandtl number," NACA TN 3986 (June 1957).

[4] Reshotko, E., "Laminar boundary-layer with heat-transfer on a cone at angle of attack in a supersonic stream," NACA TN 4152 (December 1957).

[5] Burbank, P. B. and Hodge, B. L., "Distribution of heat-transfer on a 10° cone at angles of attack from 0° to 15° for Mach numbers of 2.49 to 4.65 and a solution to the heat-transfer equation that permits complete machine calculations," NASA Memo. 6-4-59L (June 1959).

[6] Conti, R. J., "Laminar heat-transfer and pressure measurements at a Mach number of 6 on sharp and blunt 15° half-angle cones at angles of attack up to 90°, NASA TN D-962 (October 1961).

[7] Wittliff, C. E. and Wilson, M. R., "Heat-transfer to slender cones in hypersonic air flow including yaw and nose-bluntness effects," Cornell Aeronaut. Lab. Paper 61-213-1907 (June 1961).

[8] Goodwin, G., Creager, M. O., and Winkler, E. L., "Investigation of local heat-transfer and pressure drag characteristics of a yawed circular cylinder at supersonic speeds," NACA RM A55 H31 (January 1956).

[9] Sands, N. and Jack, J. R., "Preliminary heat-transfer studies on two bodies of revolution at angle of attack at a Mach number of 3.12," NACA TN 4378 (September 1958).

[10] Kopal, Z., "Tables of supersonic flow around cones of large yaw," TR 5, Dept. Elect. Eng., Mass. Inst. Tech. (1949).

[11] Kopal, Z., "Tables of supersonic flow around yawing cones," TR 3, Dept. Elect. Eng., Mass. Inst. Tech. (1947).

[12] Yen, S. M., "Investigation of laminar heat-transfer and skin friction on a slender cone at angles of attack," McDonnell Aircraft Corp. Rept. 7086 (September 1959).

MARCH 1963 AIAA JOURNAL VOL. 1, NO. 3

Some Exact Solutions for Cavitating Curvilinear Bodies

Fredric F. Ehrich*
General Electric Company, West Lynn, Mass.

A special case of cavitating flow solutions is postulated and transformed to a semi-infinite plane. The complete, exact solution then is synthesized by superposition of singularities. The solution is relevant to a general, two-parameter family of curvilinear bodies. The parameters are the flow angles at the two points of flow separation. The body reduces, in the special case, to the Rayleigh solution for a flat plate. The equations of the cavity boundaries are given in explicit form. The body form and the stagnation streamline are given as the locus of the roots of a cubic equation. Local static pressures and, hence, lift and drag, also may be calculated. The generated solutions constitute a technique involving simple computation for exact solutions of a special family of cavitating curvilinear bodies at finite angles of attack.

Nomenclature

a, b, A, B, C, E = solution constants
A, B, C, D, E, F = labels of locations in various planes
C_D = drag coefficient
C_L = lift coefficient
c = chord length
$R = (\xi^2 + \eta^2)^{1/2}$ = radial vector length in the ζ plane
r_1, r_2 = location of the singularities in the ζ plane
s_1, s_2, s_3 = location of stagnation points in the ζ plane
$w = \phi + i\psi$ = complex potential (potential function, stream function)
$z = x + iy$ = complex physical coordinate (abscissa, ordinate)
$\zeta = \xi + i\eta$ = complex half-plane coordinate (abscissa, ordinate)

Received by IAS August 10, 1962; revision received December 26, 1962.

* Engine Design Engineer, Small Aircraft Engine Department.